17th ICCRTS
"Operationalizing C2 Agility"

Paper ID 013:
On the 'Boyd-Kuramoto Model': Emergence in a Mathematical Model for Adversary C2 Systems

Topics: 6, 8

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On the ‘Boyd-Kuramoto Model’: Emergence in a Mathematical Model for Adversary C2 Systems

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Abstract

At the 13th ICCRTS I proposed that a well-known mathematical model for synchronisation on networks – the Kuramoto Model – serves as a useful paradigm for Command and Control (C2). This model places nonlinearly coupled phase oscillators at a network’s nodes. Increasing the coupling constant sees the oscillators manifest global phase synchrony. Mapping the phase to Boyd’s Observe-Orient-Decide-Act loop (or any iterative decision making process) established the relationship to C2. I generalised this to two adversaries, where ‘fighting nodes’ in a Blue network seek to ‘Act’ before adversaries in a Red network, consistent with Boyd’s model. Here, I numerically solve this ‘Boyd-Kuramoto’ model, subjecting a Blue hierarchy against a Red random network with oscillator frequencies drawn from a uniform random distribution. I represent the Intelligence-Surveillance-Reconnaissance capability of the Blue force such that a Blue node is aware of the state of a Red node’s phase across the decision-cycle while Red only has knowledge of Blue node’s state when it ‘Acts’. The model is sufficiently rich to capture surprising, but in retrospect understandable, behaviour – it displays ‘emergence’– while remaining solvable with minimal computational effort. The model allows for exploration of the degrees of freedom of a ‘rigid’ Blue force against an ‘agile’ Red adversary.

1. Introduction

In [Kalloniatis 2008] I proposed a new mathematical model to represent two adversarial Command and Control (C2) systems whose elements are engaged in iterative cycling through continuous Observe-Orient-Decide-Act (OODA) loops [Boyd 1987]. This proposal drew upon a well-known existing model in the mathematical complex systems literature that displays the phenomenon of “self-synchronisation”. Given the significance of that term in the drive to network-enable military forces through this past decade, application of this Kuramoto Model to Network Centric Warfare (NCW) seemed an obvious thing to undertake; curiously, such an application was missing in the C2 literature. Indeed, the Kuramoto model has now seen some uptake in the C2 community [Dekker 2007, van der Wal 2010, Dekker 2011] as a compact but sufficiently rich model – but always from the point of view of a single C2 system engaged in some aspect of an internal process (choosing from ‘options’ in Dekker and sensor fusion in van der Wal). As some C2 researchers have pointed out, including Boyd [1987], the adversary is often neglected when analysing a C2 system and this was the main motivation here in adapting the Kuramoto model to the case of *two rival C2 systems*. Having waited now several years for others to explore my proposal, I will address numerical solutions of it and call it the ‘Boyd-Kuramoto Model’.

The model is expressed as a set of coupled nonlinear differential equations. I put these forward as a C2 version of Lanchester’s equations for attrition, or Hughes’ equations for salvo missile combat [Hughes 2000]. In this era, proposing such a model for C2 may seem naïve. However, by any definition C2 (and we prefer that of Pigeau and McCann [2000]) is complex; no single model can be entirely valid; there is a need for diversity of models of C2. The simplest model for C2 is the beloved wiring diagram. This often extends to Social Network models to capture informal interactions. These are generally static representations. Business process models try to incorporate the time dimension, while agent based distillations offer more sophistication. However, such models are invariably tactically based and mix a rudimentary decision process (the C2 part) with kinetic activity in a representation of physical space. No single model of a complex system, such as the C2 enterprise, has universal validity [Harré 1970]. The determination of the ‘truth’ of any hypothesis for C2 requires cross-validation between a number of models. To that end, the model proposed here fills a gap across the above spectrum of existing models.

The present model combines a number of elements considered important in C2 in the context of the broader organisational theoretic literature. Boyd’s OODA loop based on his experience of US fighter pilots in the Korean War is now used across the Defence and Business environments as a simple but effective model for the iterative and cognitive aspects of decision-making. The “essence of winning”, to draw from a 5 slide set from Boyd [Boyd 1996], is “the ability to get inside other OODA loops”. An OODA cycle takes an amount of time that depends on the intrinsic ability of the individual to internally ‘process’ and the pressures on the individual to keep pace with colleagues and outpace the adversary. The property of self-synchronisation – that local interactions between agents can multiply across a coupled system in order to achieve global effects – has been highlighted with the formulation of NCW; the Kuramoto model is the mathematical representation of this *par excellence*. Of course both NCW and the Kuramoto model highlight the importance of the network of interactions. In unifying these here I achieve a mathematical instantiation of the networked OODA concept proposed by Moon, Kruzins and Calbert [2002]. By using the Kuramoto model in this way I capture another element of organisational behaviour beyond `structure’ and ‘dynamism’, namely ‘coupling’. This property of an organisation is well known to theorists such Perrow [1984], Mintzberg [1979] and others of the Contingency Theory School [Donaldson 2001]: connected nodes in a C2 structure can have quite different strengths of coupling or degrees of responsiveness to changes in their respective states. The model I propose enables an exploration of the balance between the properties of 'dynamism' (expressed in a frequency spectrum for individual performance of the OODA loop), 'structure' (expressed in a network structure) and 'coupling' (expressed in a set of constants for intra and inter C2 network coupling strengths.

A preliminary comment on synchronisation and whether or not it is a universally 'good behaviour' for a C2 system is in order here. While incoherence is never good, I do not assume here that *total* synchronisation is a good thing. I refer herein to two incomplete forms of synchronisation. 'Partial synchronisation' is where many nodes are locked together while others are subject to random behaviour with respect to the locked core, and 'sub-synchronisation' is where nodes may form into two or three internally locked clusters but each cluster moves with its own frequency [Kalloniatis 2010]. Each of these behaviours may be more useful in a C2 context than total synchronisation. I will consider each of these scenarios, which will lead to some surprises – the 'emergence' alluded to in the title. My aim nevertheless is to demonstrate the broad value of the model rather than to apply it to a specific situation. For that reason I consider simplified representation of contemporary C2 systems: a tree hierarchy and a random network. Western military forces are no more rigid one-dimensional hierarchies than terrorist/insurgent networks are purely random. The former already incorporate informal links [Ali 2011] and the latter are known to have elements of a hierarchy [Memon, Larsen, Hicks and Harkiolakis 2008]. Thus my aim is not to validate a model in some specific context and to extract recommendations for that situation but to demonstrate the broad applicability of the 'Boyd-Kuramoto Model'.

The paper is structured as follows. Section 2 describes the model, while Section 3 defines Measures of Effectiveness for intra- and inter-C2 performance. Section 4 presents some extreme scenarios of two forces respectively in the incoherent and fully self-synchronised states as an aid to understanding of the basic patterns of behaviour. Section 5 explores some sub-synchronised behaviours and demonstrates unexpected behaviours that, in retrospect, can be understood. The paper closes with suggestions for future enhancements of the model.

2. The Boyd-Kuramoto Model

The literature on self-synchronisation in mathematically encoded cooperative systems is vast, going back to Wiener [1961] and Winfree [1967] and scattered across mathematical, physical, biological and computational scientific journals. The basic idea of such models is that linking up nodes that individually undergo cyclic behaviour can lead to a mass effect whereby a large part of the system locks itself into a collective cyclic behaviour. This is legitimately called self-synchronisation because it is a consequence of many local interactions and not the manipulation of the system by a master-controller. It was Kuramoto [1984] who succeeded in distilling the bare essentials of such models into the first order differential equation:

$$\dot{\beta}_i = \omega_i + \sigma \sum_{j=1}^{N} \sin(\beta_j - \beta_i) . \quad (1)$$

Here β_i represents a time-dependent *phase* associated with node *i* of a complete network of *N* nodes, $\dot{\beta}_i$ is the angular rotation speed via the derivative of the phase with respect to time t, ω_i represents a 'natural' or 'intrinsic' frequency, usually randomly chosen from a statistical distribution, and σ is a coupling constant. The role of β_i as a phase is seen when it is reinserted in the complex variable

$$\chi_i = e^{i\beta_i} . \qquad (2)$$

A general network is introduced straightforwardly using the adjacency matrix A_{ij} whose elements take value one if a link (or edge) exists between nodes *i* and *j* and are zero otherwise; for simplicity we remain within the bounds of undirected graphs, though further generalisations are possible. The governing time evolution equation is then:

$$\dot{\beta}_i = \omega_i + \sigma \sum_{j=1}^{N} A_{ij} \sin(\beta_j - \beta_i). \qquad (3)$$

The behaviour of the system can be visualised as points moving about the unit circle as in Figure 1. At any point in time each oscillator will be represented by a point on that circle.

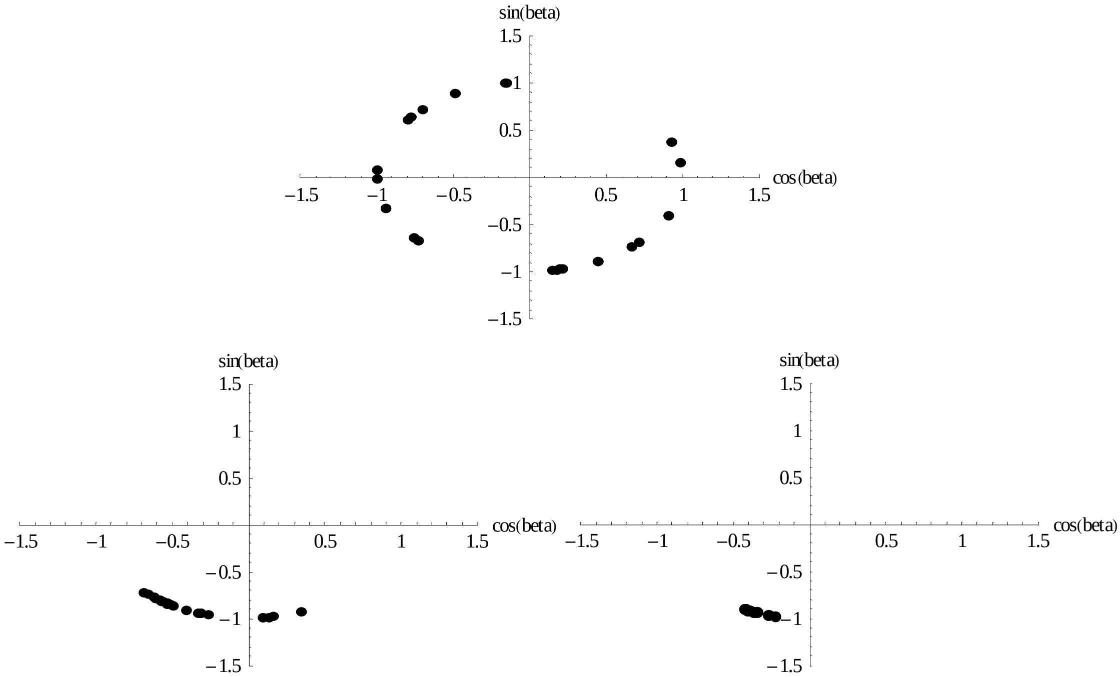

Figure 1 Visualising the individual oscillators as points distributed on the unit circle at a snapshot in time for different values of coupling: weak coupling (top), strong coupling (bottom left) and very strong coupling (bottom right).

For weak coupling the points will be randomly distributed around the circle (Figure 1, top), given the random individual frequencies. For strong coupling the points move with the same angular speed and group increasingly together as the coupling is increased (Figure 1, bottom left and right).

Translating this to the C2 context, the phase $\beta_i(t)$ represents the point in a continuous decision (or OODA) cycle of agent i at some time *t*. The network represents the C2 structure itself: the relationships of agents who need to mutually adjust their individual decision cycles. The coupling σ is a somewhat more abstract concept but can be seen as how `quickly' one agent should adjust their progress through the decision cycle given a change in the progress by any other. The 2π-periodicity of the sine function is appropriate in that it locally synchronises decision cycles within the 'current phase'. The frequency ω_i is how many decision cycles per unit time can be achieved by agent i. This is chosen from a random distribution, representing the underlying heterogeneity between individual decision makers in the C2 system. Training and discipline can narrow that distribution; namely, introducing more homogeneity in the population of decision makers. But the intent is nonetheless to retain some degree of heterogeneity. Moreover, one does not have the luxury of 'managing' that heterogeneity: the C2 system is not designed with individuals of certain frequencies placed deliberately at certain nodes.

Certainly in the NCW literature, such as [Alberts and Hayes 2007] and references therein, the desired self-synchronisation is applied to *activity* in the external environment. I am proposing that the precursor to this is synchronisation of *decision cycles* and therein mapping the phase of the Kuramoto model to the decision cycle; another implementation of the Kuramoto model is possible at the level of activity and is that used in [Dekker 2007, 2011]. These two options are not very far apart: a decision cycle in a context such as a headquarters will very often leave a trail of external artefacts (draft documents, emails, chat or verbal communication) that indicate to the stage of OODA of a unit or individual; these artefacts are thus points of reference for another in the same organisation in synchronising their cycle. In other words, even the cognitive stages of Observe-Orient-Decide involve some form of external activity, is a social enterprise, when one steps beyond Boyd's original application to the isolated fighter pilot alone in the cockpit.

I now turn to the case of Blue-on-Red interactions, the regime of Boyd's original OODA concept. I now associate β_i with the point in a continuous decision loop of 'agents' of the Blue C2 system, ρ_i, with the OODA loop of Red agents, and intrinsic frequencies ω_i and ν_i respectively. Let B_{ij} and R_{ij} represent the adjacency matrices of the *decoupled* Blue and Red networks, of size N_B and N_R respectively. The interactions between Blue and Red systems are represented by the adjacency matrix M_{ij} with $N_{BR} < N_B + N_R$ nodes. Finally let $\sigma_B, \sigma_R, \varsigma_{BR}$ and ς_{RB} represent coupling constants for intra-Blue, intra-Red, Blue-to-Red and Red-to-Blue interactions. The model can now be given in terms of the coupled differential equations:

$$\begin{aligned}\dot{\beta}_i &= \omega_i + \sigma_B \sum_{j=1}^{N_B} B_{ij} \sin(\beta_j - \beta_i) + \varsigma_{BR} \sum_{j=1}^{N_{BR}} M_{ij} F(\rho_j - \beta_i) \\ \dot{\rho}_i &= \nu_i + \sigma_R \sum_{j=1}^{N_B} R_{ij} \sin(\rho_j - \rho_i) + \varsigma_{RB} \sum_{j=1}^{N_{BR}} M_{ij} G(\beta_j - \rho_i).\end{aligned} \qquad (4)$$

Implicit to the intra-C2 interactions here (the sine functions) is an assumption of a node's complete knowledge about the state of a connected partner at any moment in time (in other words, connection in the unweighted intra-C2 graph is based on complete knowledge). This is a reasonable aspiration within a C2 system – thanks to the artefacts of the cognitive process mentioned above – which is not generalisable to the adversary. Thus the inter-C2 interaction functions, F and G, are different. I refer to these as 'ISR functions' for the Blue-Red and Red-Blue interactions as they reflect the respective Intelligence-Surveillance-Reconnaissance (ISR) capabilities of the two forces. I shall assume an asymmetry, that the Blue force has superior ISR over Red (contrastingly, later when I specify the networks, I shall take Red to have a 'superior' internal network). In the most extreme case, the only information a Red node may have of the state of a Blue adversary is when Blue performs a visible activity or leaves a visible artefact in the external environment, namely 'acts'. The time at which this occurs would be the point at which Red would synchronise their 'observe' activity: Red observes Blue's instantaneous action and thus synchronises to that point. Thus sine functions (as in the intra-C2 systems) may be used again (to reflect local synchronisation) but only at a specific state of Blue at the point in time at which it occurs. Assigning the four OODA stages to the four quadrants of the unit circle, with

‘Observe’ coinciding with the first quadrant, then the point at which Red should synchronise to Blue corresponds to some phase angle κ in the fourth quadrant in Figure 2.

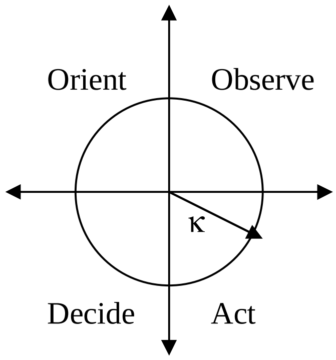

Figure 2 Choosing a point in the OODA loop at which an Action is visible to an adversary

Thus, for this extreme case, Red’s ISR function (of Blue) is

$$G(\beta_j(t)-\rho_i(t)) = \sin(\beta_j(t)-\rho_i(t))\delta(\beta_j(t)-\kappa)$$

with a Dirac delta function capturing the instantaneity of Blue ‘observing’ Red ‘act’. The ‘instantaneity’ is perhaps too strong and should be softened by a bell-shaped curve indicating that the visibility of Blue’s action involves some build-up towards a peak of activity (with output ‘1’) and a fall-off afterwards. Red has an opportunity to observe, and synchronise within, some extended period around this activity (Figure 3). Thus I take for Red’s ISR function

$$G(\beta_j(t)-\rho_i(t)) = \sin(\beta_j(t)-\rho_i(t))\exp(-(\beta_j(t)-\kappa)^2/2s^2), \qquad (5)$$

where s determines the width of the curve.

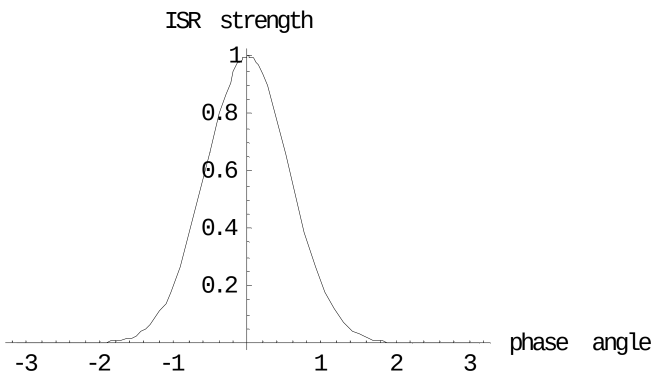

Figure 3 Bell-shaped smearing of the ISR function about an ‘Action’ occurring at phase angle zero with width $s = \sqrt{\pi}$

The contrasting extreme case to ‘no-knowledge’ of the adversary is that Blue has perfect ISR of Red: the exponential in Eq. (5) is replaced by a ‘1’ everywhere. Furthermore, Blue’s doctrine follows Boyd, whereby an advantage of Blue over Red is to be gained by “getting inside the adversary’s OODA loop”. Here I make a deliberate choice: Boyd’s formulation is not mathematically specific. Rotating through the OODA loop faster in itself will not achieve success in the instantiation of the model I am developing: eventually Blue will lap Red and turn out to be behind Red’s decision cycle. I will therefore explicitly say that “getting inside” means *synchronising a step ahead of the adversary*. This is achieved by gaining insight into the adversary’s internal processes in order to anticipate their external activities. This can now be mathematically represented with an additional displacement λ in the argument of the sine function (similar to the parameter κ). Thus Blue's ISR function is taken to be:

$$F(\rho_j(t) - \beta_i(t)) = \sin(\rho_j(t) + \lambda - \beta_i(t)). \qquad (6)$$

There are many further modifications that can be made here: using an exponential smearing like Eq. (5) also for Blue, giving Blue only intermittent intelligence at certain points of Red's internal 'observe-orient-decide' stages, using noise functions on those points. The mathematically inclined reader will see the straightforward generalisations that are possible here. For the purpose of this first look at its utility, the model is completely specified by Eqs. (4,5) and (6).

3. Measures of Performance

I now need to specify what quantities should be computed from the solutions to the differential equations. For a measure of synchronisation within a network, Kuramoto's order parameter, applied separately to the Blue and Red networks respectively, is appropriate:

$$r(t)e^{i\Psi(t)} = \frac{1}{N}\sum_i \chi_i(t). \qquad (7)$$

The sum is taken over the *N* nodes in the respective network. Values of r close to 1 over time indicate high synchronisation. For the Blue and Red systems, specifically, the synchronisation parameter will be denoted r_B and r_R respectively. When the system is highly synchronised, the angle Ψ represents the 'collective phase' of the system (or centroid of the points for each oscillator on the unit circle). More directly, this angle is calculated by

$$\Psi = \mathrm{ArcTan}\left[\frac{\sum_i \sin\theta_i}{\sum_i \cos\theta_i}\right]. \quad (8)$$

Thus Ψ_B would be the collective phase for the Blue nodes, where angles β are used to calculate Eq. (8), and Ψ_R corresponds to the Red collective phase based on angles ρ used in Eq. (8).

One measure of the adversarial performance can be extracted directly from Kuramoto's order parameter: the difference of the two collective phases:

$$\Delta\Psi(t) = \Psi_B(t) - \Psi_R(t). \qquad (9)$$

A positive value of $\Delta\Psi$ at some point in time *t* means that *collectively* the Blue C2 system is ahead of the Red C2 system within the current phase. However, such a collective measure may not fairly reflect the performance of the 'fighting' or 'tactical nodes' directly in contact with the adversary. Thus, another measure is the difference of collective phases of the fighting nodes, $\Delta\Psi_{\mathrm{tactical}}$, where Eq. (9) is used but only considering in the sum in Eq. (7) those nodes directly connected to an adversary. Either way, as mentioned, Ψ really only has meaning if both networks are exhibiting some degree of internal synchronisation. Therefore, a more direct measure is also worthy of consideration: the average difference in phases of the tactical nodes:

$$\Delta_{BR}(t) = \frac{1}{N_{BR}}\sum_{i=1}^{N_{BR}}\left(\beta_i(t) - \rho_i(t)\right). \qquad (10)$$

I show below that all of $\Delta\Psi$, $\Delta\Psi_{\mathrm{tactical}}$ and Δ_{BR} provide insight into the success, or otherwise, of the Blue network in getting inside the OODA loop of Red. Indeed, both plotting of these measures as functions of time and computing time averages will prove useful.

4. Basic Scenarios

As hinted at the outset, I consider a Blue hierarchical network in competition with a random Red network. The Blue structure is a simplification of a traditional military force operating within a framework of accountability and legally delegated authority. The Red force may be an insurgency network, not bound by the same legal frameworks as the Blue force, and being more interconnected due to family and tribal relationships. As mentioned at the outset, both are a caricature of the real world. Both networks consist of 21 nodes. Thus the Blue network is the classic tree structure (a 'quaternary tree') of Figure 4. The Red network is a random graph where links between nodes occur with probability of 0.4, as in the right hand panel of Figure 4.

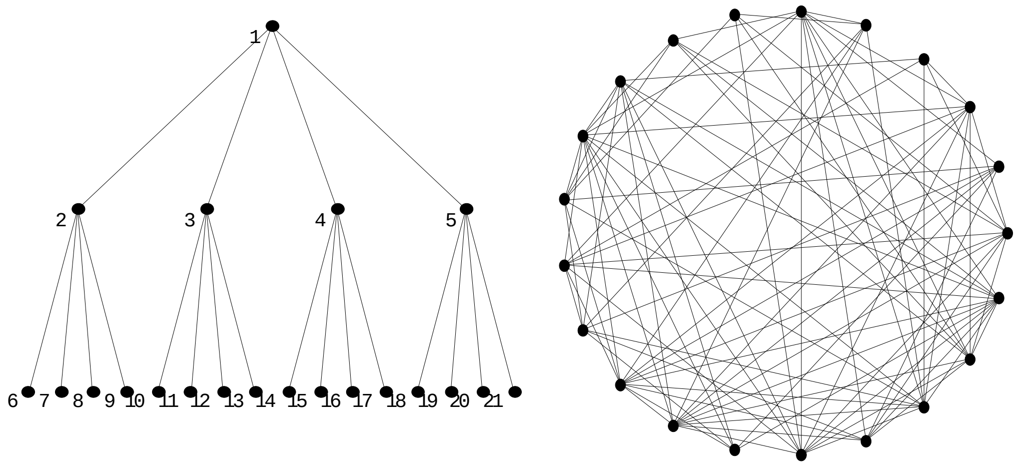

Figure 4 The tree hierarchy of the Blue C2 network (left) and the random graph of the Red C2 network

The Blue-Red cross interactions involve each of the 16 'tactical' nodes in the Blue network linked to a single random node in the Red network. In other words, the Blue side clearly demarcates who is a 'fighter', who is a team leader or conduit of information, and who is the overall commander. The Red side has no such delineation: any 16 of the 21 nodes may be a fighter, according to chance encounter with the adversary; after that, anyone may be a conduit.

The intrinsic frequencies are drawn from a uniform distribution between zero and one, so that both sides are attempting to conduct their decision cycles within the same broad time period (frequency of one may mean one decision per 24 hour period, but this factor may be scaled out) but there is heterogeneity in the speed of individual nodes. This means that, although the hierarchy has a number of echelons, in this specific instantiation of the model they do not work within nested cycles as may be the case for a unit headquarters in relation to a higher operational headquarters; for example the former may plan and act within a 24 hour cycle while the latter plans over periods of weeks to months encapsulating many cycles of its subordinate unit. An example of a hierarchy such as in Figure 4 with all layers working to the same broad cycle is the XIX Corps Headquarters of Heinz Guderian in the attack on French forces through the Low Countries in 1940. As documented in his memoirs, and translated in [Fitzgibbon 2001], Guderian's headquarters would issue orders to subordinate units in the same 24 hour cycle of their activity thus succeeding in operating inside the OODA loop of opposing French forces.

The Blue network has complete ISR on Red and, in my instantiation of Boyd's doctrine, seeks to stay one quarter of a cycle ahead of Red. Thus, we have $\lambda = \pi / 4 \approx 0.7854$. Red, on the other hand only has visibility of Blue's actions consistent with Figure 3.

The system of equations are not readily analytically solvable due to the property that the interaction in the Kuramoto model is a (sine) function of differences of phases, not a difference of functions of the phases. This simple distinction means that the approach, using Lyapunov theory, of [Chen and Lu 2008] to a similar problem of two interacting networks cannot work here. I therefore numerically solve Eqs. (4) with `Mathematica` for a time interval of 200 units using its 'NDSolve' function for solving coupled differential equations. The initial conditions for both Blue and Red phases are drawn from a uniform random distribution between $-\pi/2$ to $\pi/2$. A maximum number of steps of 10000 has proved sufficient for solving in the incoherent regime, which is most taxing due to the zig-zag nature of most of the variables. The computation takes less than one minute on a desktop computer; calculation, with the numerical solutions, of the measures discussed in this paper is a matter of a few minutes. Normally one would seek to average measures over a number of instances from the frequency distribution. However, this may smear out some subtler aspects of behaviour, to be discussed later. Therefore in this paper I present the results of specific instances but these show otherwise 'typical' behaviour.

To obtain an initial sense of typical behaviours I present here two extreme cases: the C2 networks internally poorly coupled but Blue strongly coupled across to Red; and the two networks strongly internally coupled but weakly cross-coupled. Thus the first case represents two forces whose individual C2 is 'poor' but with Blue more focused on its engagement with its adversary. The second represents forces who are overly concerned with their internal C2 but with little regard for the adversary.

4.1 Poor C2-Strong Adversarial Engagement

The couplings for this case are chosen to be

$$\begin{array}{ll} \sigma_B = 0 & \varsigma_{BR} = 30 \\ \sigma_R = 0 & \varsigma_{RB} = 0 \end{array}. \qquad (11)$$

Examining the variables r_B and r_R we see the typical zig-zag pattern for the incoherent regime in both networks in Figure 5. However, there is some degree of coincidence between peaks of the two curves consistent with the property that Blue tactical nodes are strongly engaged with their adversaries. This could be verified by performing a correlation analysis between the two time-series, which we forego here for the sake of brevity.

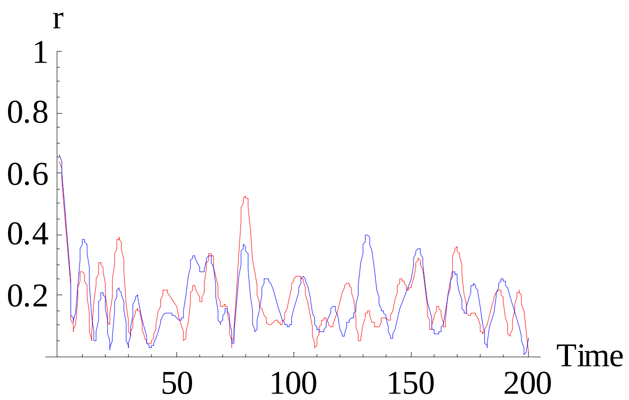

Figure 5 The synchronisation parameter as a function of time for the two networks for the case Eq. (11)

The success of the Blue tactical nodes in keeping $\pi/4 \approx 0.7854$ ahead of their adjacent Red nodes is seen in the average phase difference for those nodes in Figure 6.

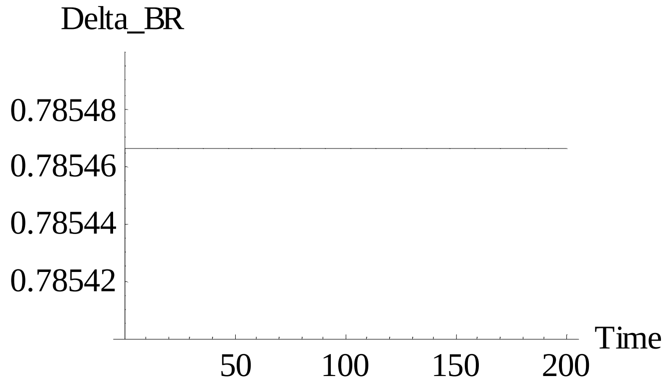

Figure 6 The average phase difference for the tactical Blue and Red nodes, as a function of time for the case Eq. (11)

The collective phase here is not a useful variable since the two systems are internally incoherent. However, for completeness, the result in this case is shown in Figure 7. The behaviour is more incoherent due both to the inclusion of all nodes in the networks and because the definition of Eq. (7) is $\bmod \pi$. However, the basic feature remains that the Blue force, even by this coarse measure, is able to remain $\pi / 4$ ahead of Red.

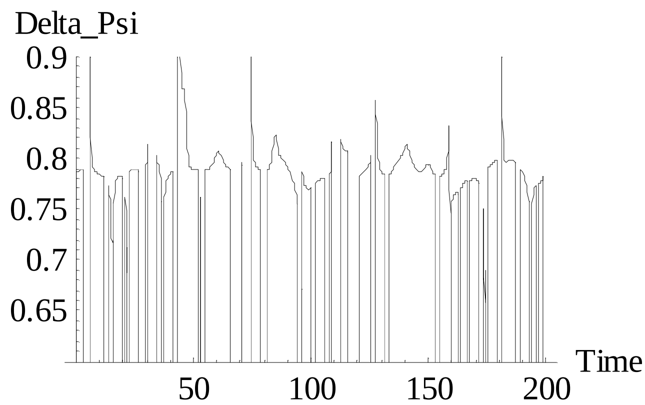

Figure 7 Difference between collective phases for the two C2 networks for the case Eq. (11)

4.2 Strong C2-Poor Adversarial Engagement
Now I consider the coupling regime

$$\begin{array}{ll} \sigma_B = 0.8 & \varsigma_{BR} = 0 \\ \sigma_R = 0.15 & \varsigma_{RB} = 0 \end{array}. \quad (12)$$

The synchronisation parameters for the two networks are given in Figure 8 showing the typical behaviour for the strongly coherent regime, with both r_B and r_R stabilizing at a non-zero value. The Red network is better synchronised than Blue: more of the individual oscillators have locked into the collective mode. However, the internal couplings have been selected to be close to the threshold for this behaviour. Thus the hierarchy requires stronger internal coupling than the random network in order to achieve phase synchronisation, essentially because of the better connectivity within the random network (for example, smaller average distance [Dekker 2007]).

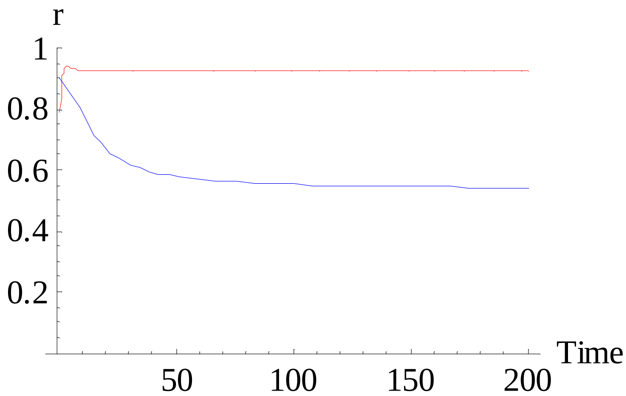

Figure 8 Synchronisation for the two networks for the case Eq. (12)

It so happens that in this numerical run the average frequency of the Red nodes is $\overline{\omega} = 0.61$ while for Blue it is $\overline{\omega} = 0.56$. This means that the collective phase of the Red network should be advancing faster than the collective phase for the Blue network; Red will lap Blue, not due to any 'intent' but simply due to its own internal dynamics. Because of the better synchronisation in both networks, the collective phase is a more insightful variable. For this case the behaviour is seen in Figure 9. Apart from the jumps due to the $\bmod \pi$ ambiguity in Eq. (7), a negatively sloped linear behaviour is seen consistent with Red out-pacing Blue.

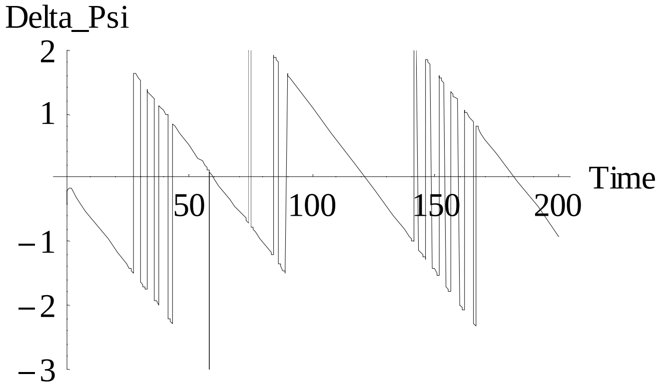

Figure 9 The difference between collective phases for the two networks for the case Eq. (12)

(For runs where the random frequencies deliver an average Blue frequency greater than that of Red I have checked that the linear slope is positive.) This lapping phenomenon can also be seen in the average phase difference, Figure 10, but now not polluted by the phase jumps: the average phase for Blue runs behind that of Red with a straightforward linear dependence.

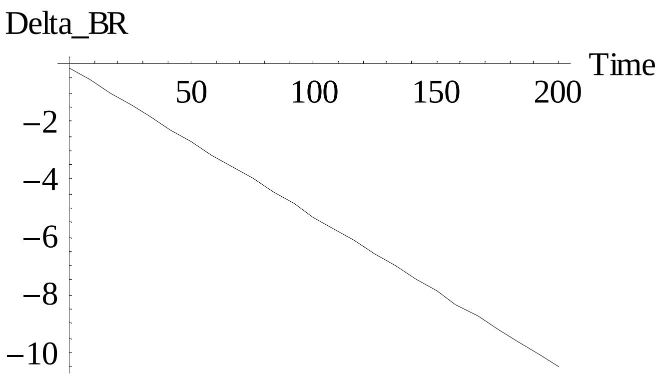

Figure 10 The average phase difference for the tactical nodes for the case Eq. (12)

Having set the scene for 'expected behaviours', I am now positioned to explore the interesting cases where couplings fall into intermediate regimes and the two C2 systems are caught between their internal dynamics and their engagement with the adversary.

5. Emergent behaviours

For the 'layman', emergence is the property of dynamical systems to 'surprise' – to exhibit unexpected behaviours. Above, I have shown a number of foreseeable behaviours. What other behaviours might the reader now anticipate? To make this more precise, I adopt Laughlin's [2005] definition of emergence as: system qualities or behaviours that are not reducible to the system components but arise from their interactions. In this case there are a number of layers explicitly built into the system *design*: 1) the individual oscillators at the nodes of each network, 2) the two networks as entities unto themselves, and 3) the collective system of Blue and Red interacting networks. I shall be primarily interested in emergence across these layers; namely, behaviours that are not reducible to one of these three layers.

These ideas can be made mathematically concrete. Emergence in dynamical systems is also associated with an intermediate region between order and disorder, stability and instability or the 'Edge of Chaos'. Formally, this means the existence of fixed points in multi-dimensional phase space, in the vicinity of which some trajectories neither exponentially converge back to the point (Lyapunov stable) nor diverge away (Lyapunov unstable) but follow power-law dependence on time. Mixed in with more standard stable and unstable directions, this gives rise to forms of patterned behaviour through collective degrees of freedom. My colleague Richard Taylor has shown that there are thresholds for more types of fixed points in the equal frequency Kuramoto model than just 'globally phase synchronised' [Taylor 2012]. In the Kuramoto model with non-equal frequencies, I have also identified such fixed points [Kalloniatis 2010]: for many classes of networks there is an intermediate range of coupling where nodes have formed a small number (two to three) of clusters, within which oscillators are locked to a common frequency, but across which there remains incoherence; a further increase in coupling tips these clusters into forming a single overall cluster (I have shown that these behaviours occur in a regime of vanishing real parts of Lyapunov exponents). The system of many oscillators *may* devolve to a two or three body system of effective modes, the internally locked clusters, sufficient to give rise to structured behaviour. The membership of these clusters depends on the vagaries of how the frequencies are distributed: oscillators with nearly identical frequencies placed at adjacent nodes will tend to cluster. This clustering as an intermediate regime is illustrated in a series of parametric plots for three different values of the coupling σ_B in Figure 11.

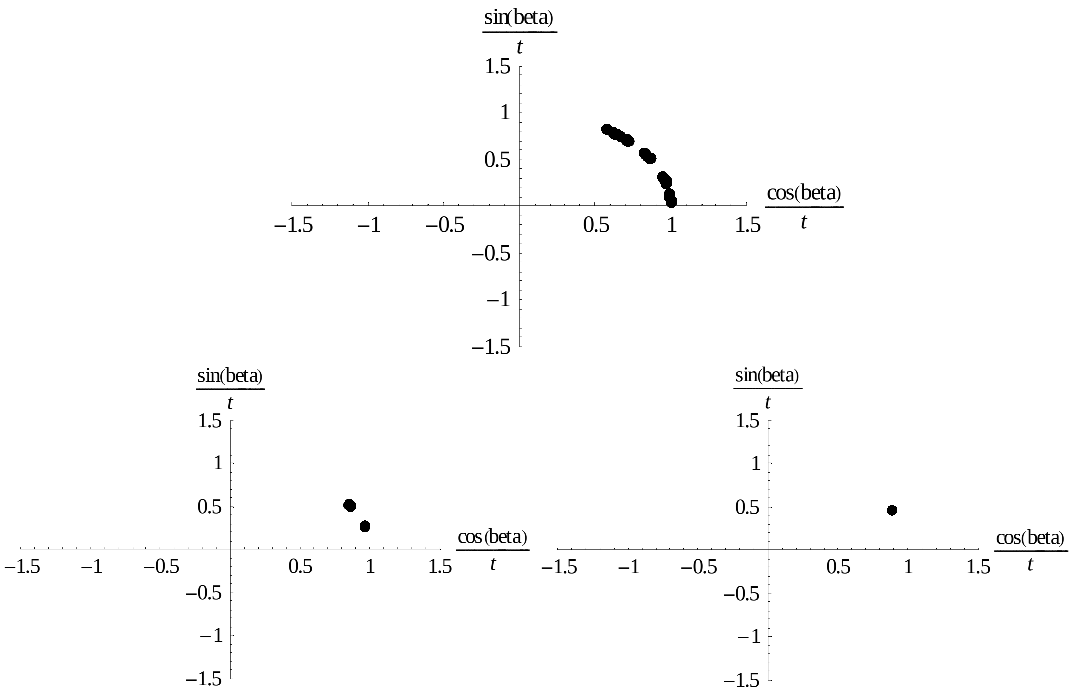

Figure 11 Parametric plots over the time duration of cosine and sine of the Blue phases divided by time for weak (top), intermediate (bottom left), and strong (bottom right) coupling; respectively these show incoherence (top), intermediate clustering (bottom left) and complete phase locking (bottom right).

Unlike Figure 1, which represents a snapshot in time, I am now plotting as a series of points the cosine and sine of the phase of a Blue oscillator *for each moment in time*. For any single point I would obtain a circular track. Dividing the cosine/sine *by the time* brings this track, for zero coupling, to a single point consistent with the motion of an individual oscillator about the unit circle being largely uniform in time. Plotting this for all oscillators, for zero coupling, gives a distribution of points lying on the circle (Figure 11, top); unlike the first case in Figure 1, the points here are not spread over the entire circle because dividing by the time here exposes the individual oscillator frequencies, which are selected from the range [0,1]. Now, as coupling strength increases there is a transition from the multiple points to one single point, corresponding to all oscillators locked to the same phase moving with the average frequency (Figure 11, bottom right). In between these extremes is a state of two independent clusters (Figure 11, bottom left). This intermediate level clustering gives rise to *cyclic* behaviour of the order parameter *r*. Two examples of this are shown in Figure 12, respectively for the Blue and Red systems, with couplings

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 0 \\ \sigma_R = 0.075 & \varsigma_{RB} = 0 \end{array} . \quad (13)$$

For a given network structure, within a range of couplings (intermediate between strong, for coherence, and weak for incoherence) cyclic behaviour of the order parameter will occur with probability almost one for any configuration of frequencies from a selected distribution. On the other hand, the *periods* of the cycles are sensitive to the particular instance of frequencies, whereby the aforementioned clusters consist of nodes both related to each other by connectivity (according to the network) but also according to closeness in frequency. Averaging over a sample will smear this behaviour out, for example giving time-averaged values of $\langle r_B \rangle \approx 0.8$ and $\langle r_R \rangle \approx 0.6$. This is the reason I present specific instances. The patterns in Figure 12 are consistent with their being two clusters in each of the Blue and Red systems. The presence of a third can create additional ripples in the otherwise sinusoidal behaviour. Properties of the periodicity and Lyapunov exponents for this behaviour have been analysed in [Kalloniatis 2010].

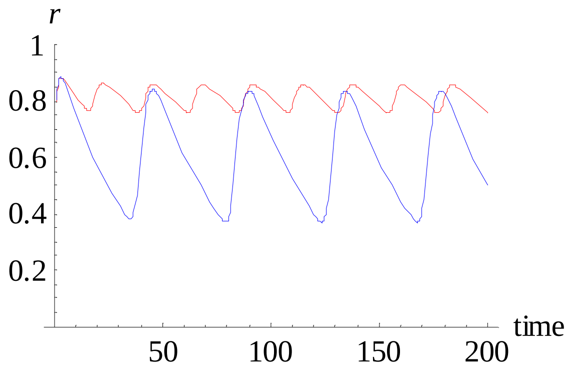

Figure 12 Two examples of cyclic-self-synchronisation

I refer to this behaviour as cyclic-self-synchronisation.

This intermediate phenomenon has a straightforward C2 interpretation: within the C2 system, two or three sub-structures form that lock out-of-phase with each other or move in and out-of-phase. This may be a desired behaviour: teams within the C2 system may not be required to be rigidly in-phase due to operating in different time-zones (say, an in-theatre deployed team maintaining links back to a control centre in an operational or strategic headquarters; two teams working on relatively decoupled aspects of an operation). The behaviour may be undesirable: a cluster of members of the C2 system who are never able to 'get on the same page' as the rest, who have barely understood the mission in a planning activity when the rest are already developing courses-of-action. In such cases, the full human capital of a C2 team is never brought to bear at the same time on the situation.

In the following, I examine variations from this behaviour as the coupling between Blue and Red is varied, keeping fixed the frequency samples as generated the patterns in Figure 12. It suffices to say that the average frequencies arising here are $\overline{\omega}_B = 0.53, \overline{\omega}_R = 0.50$. Thus, for the case Eq. (13), with no cross-coupling Blue should lap Red despite their respective cyclic-self-synchronous internal behaviour. This is evident in Figure 13, particularly Δ_{BR} (right panel).

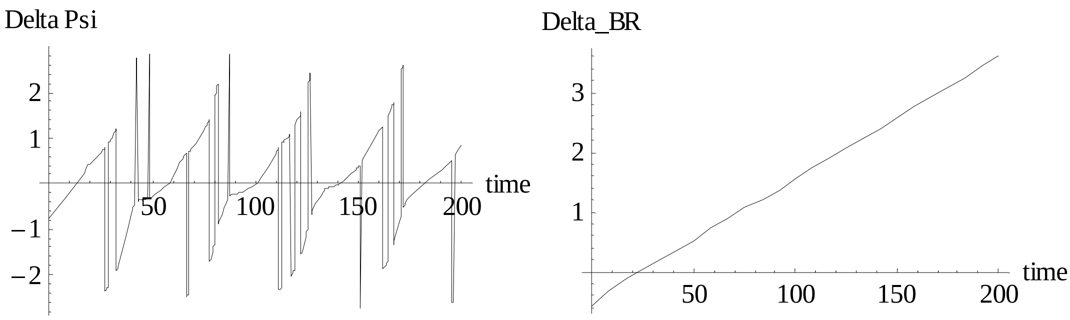

Figure 13 The two cross-system measures for the case Eq. (13)

5.1 Blue cyclic-self-synchronous vs Red cyclic-self-synchronous

I consider first the case of 'weak coupling' of Blue to Red – in the sense that Blue is coupled to Red more weakly than it couples within itself:

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 0.1 \\ \sigma_R = 0.075 & \varsigma_{RB} = 0 \end{array} . \quad (14)$$

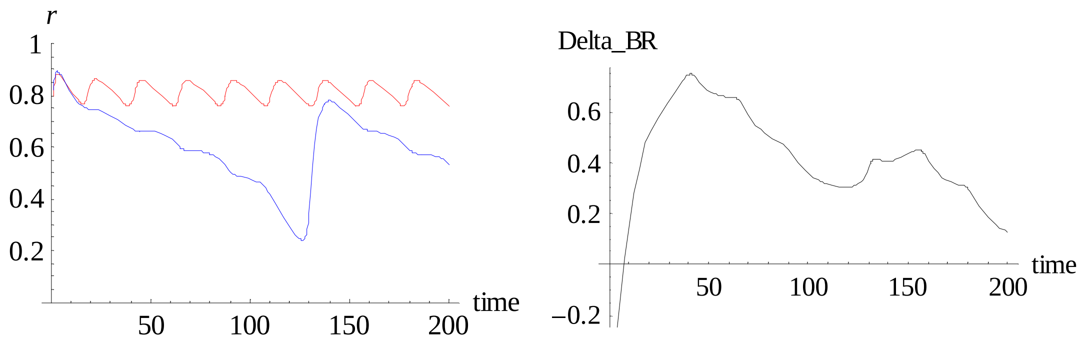

Figure 14 Behaviours of the system for case Eq. (14)

Red is unchanged (it does not 'see' Blue): the Red curve in the left panel of Figure 14 is the same as that in Figure 12. It is evident that Blue is 'struggling': its own internal synchronisation has worsened in the sense that the period of the cyclic-self-synchronisation has increased (compared to the period in the Blue curve of Figure 12 – I have checked that over longer periods of time the pattern is repeated). However, there are hints of a faster sub-frequency, which corresponds to the Blue fighting nodes. This in turn matches the period in the Red C2 system. However, looking at the average frequency difference between the two sets of tactical nodes (right panel, Figure 14) it is clear that the Blue 'fighters' are failing to stay that step ahead of their adversaries; they are caught between being locked into their C2 structure and being responsive to the enemy. But it is even more complex than that with the adversary also consisting of two clusters.

Now I increase the Blue-Red coupling:

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 0.8 \\ \sigma_R = 0.075 & \varsigma_{RB} = 0 \end{array} . \quad (15)$$

Thus Blue tactical nodes are coupled to their adversary more strongly than they are coupled to each other and their other C2 partners. The measures for this case are combined in Figure 15.

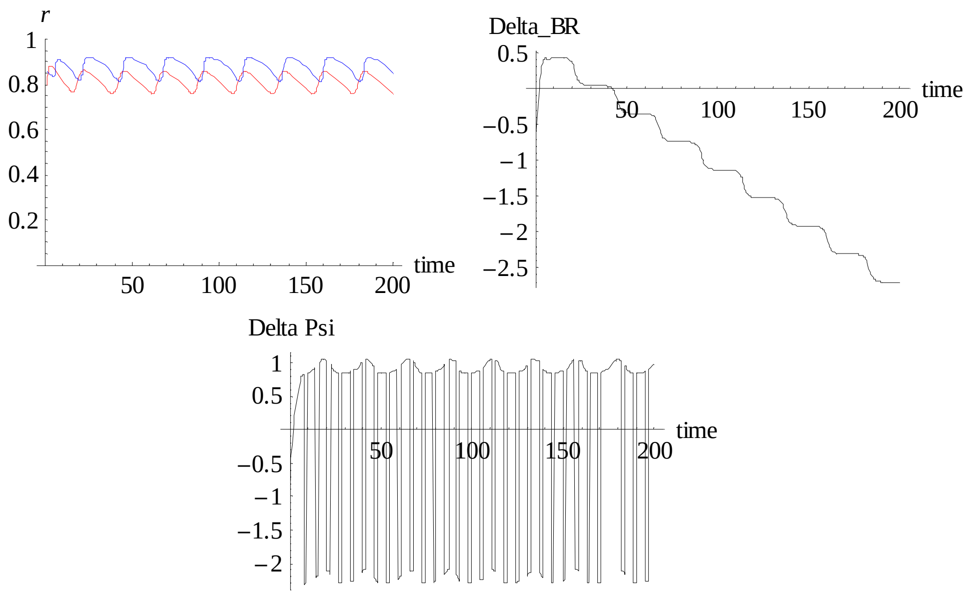

Figure 15 Measures for the case Eq. (15)

The top left panel of Figure 15 suggests that Blue is much more successful now with its cyclic-self-synchronous behaviour almost mirroring that of Red. From the perspective of the difference of the collective angles, bottom panel of Figure 15, the same picture emerges. However the average difference of the tactical node phases tells us a subtler story. There are indeed periods of time where the clusters of Blue are staying locked with respect to the clusters of Red – the flat parts of the curve in the top right panel of Figure 15 – but then they 'slip' and lock again. The overall slope of the descending stairs is indicative of Red lapping Blue.

Now I increase the coupling of Blue to Red even further, making it significantly greater than that for its internal network:

$$\begin{matrix} \sigma_B = 0.6 & \varsigma_{BR} = 2 \\ \sigma_R = 0.075 & \varsigma_{RB} = 0 \end{matrix}. \quad (16)$$

The key behaviours are plotted in Figure 16. Finally, Blue is proving successful. The left panel of Figure 16 suggests Blue is completely self-synchronous but for very low amplitude cycles that appear to mirror the periodicity of Red. The right panel shows further how successfully the tactical nodes stay close to their intended fixed $\pi/4 \approx 0.7854$ phase ahead of their adversaries. However, there remains a cost at being attached to the Blue system with the same coupling as the other Blue nodes, with minor oscillations around the ideal point. By now a consistent picture is seen between Δ_{BR} and $\Delta\Psi$. The former offers a cleaner insight into the behaviour of the system. I therefore forego the corresponding plots for $\Delta\Psi$ for the remainder of the paper.

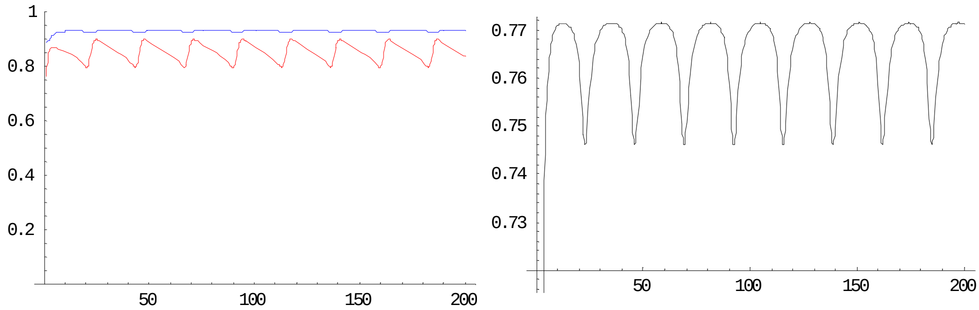

Figure 16 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (16)

5.2 Full Blue self-synchronicity vs Red cyclic-self-synchronicity

Now I examine the case of Blue internally coupled such that, left to itself, it fully synchronises, but Red is in a state of cyclic-self-synchronisation:

$$\begin{matrix} \sigma_B = 0.6 & \varsigma_{BR} = 0 \\ \sigma_R = 0.08 & \varsigma_{RB} = 0 \end{matrix} . \quad (17)$$

I use, again, a different selection of frequencies, now with $\overline{\omega}_B = 0.53, \overline{\omega}_R = 0.48$. The basic measures are show in Figure 17.

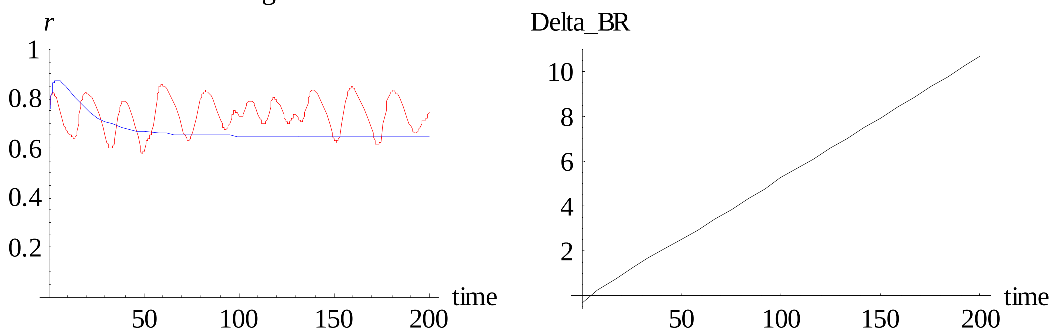

Figure 17 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (17)

Now I begin to increase the coupling of Blue to Red to be ‘weak’, less than the internal coupling of Blue:

$$\begin{matrix} \sigma_B = 0.6 & \varsigma_{BR} = 0.2 \\ \sigma_R = 0.08 & \varsigma_{RB} = 0 \end{matrix} . \quad (18)$$

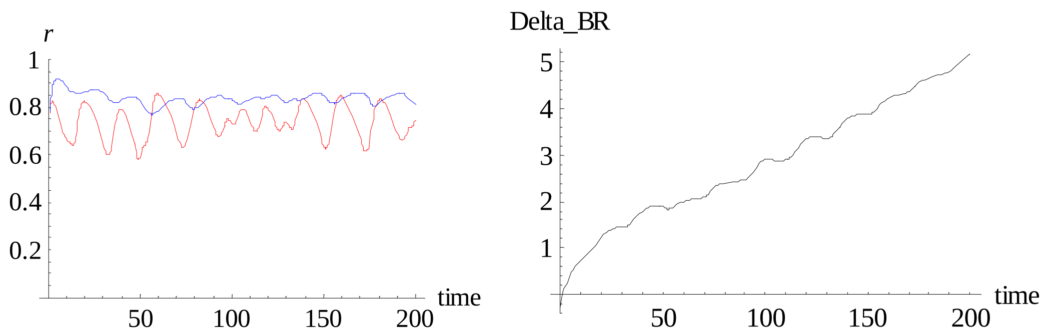

Figure 18 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (18)

Blue begins to respond to the cyclic fluctuations of Red, but quite poorly, indeed worse than in the cases for Figure 14 and Figure 15: the strength of Blue's internal coupling has to be overcome. Levelling the two couplings out,

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 0.6 \\ \sigma_R = 0.08 & \varsigma_{RB} = 0 \end{array}, \quad (19)$$

leads to the performance in Figure 19. Initially Blue appears to be successful in locking into its desired point ahead of Red, up to some fluctuations: the r-parameter of the Blue force begins to reflect the cyclic patterns of Red, and Δ_{BR} is initially fluctuating around 0.87. However, over time the behaviour begins to 'slip away', with a step pattern emerging across longer periods of time (superimposed with small oscillations), much like the behaviour in Figure 15.

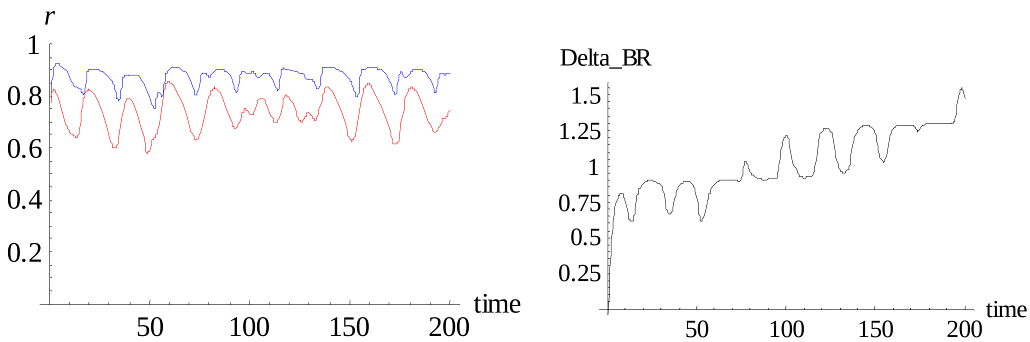

Figure 19 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (19)

Now I increase the coupling of Blue to Red even further:

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 2 \\ \sigma_R = 0.08 & \varsigma_{RB} = 0 \end{array}. \quad (20)$$

The performance measures are shown in Figure 20. This is close to the situation in Figure 16: small oscillations by Blue close to the intended point ahead of Red. However, this point is not precisely the desired point; the case of Eq. (16) may be deemed to be more 'successful'.

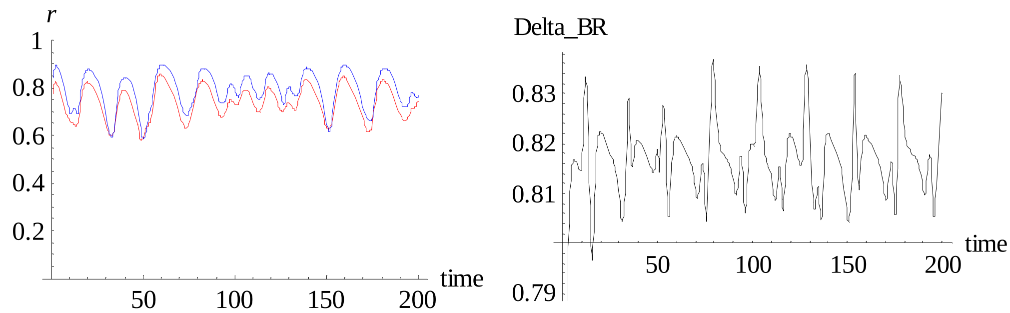

Figure 20 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (20)

Now I weaken the asymmetry between Blue and Red: I allow Red some knowledge of Blue's OODA loop through visibility of external artefacts in Blue's 'Actions'. I therefore choose the parameters:

$$\begin{array}{ll} \sigma_B = 0.6 & \varsigma_{BR} = 2 \\ \sigma_R = 0.08 & \varsigma_{RB} = 1 \end{array}. \quad (21)$$

Thus Red is adapting to Blue, just as Blue has been adapting to Red. Note that the strength of Red's coupling to Blue is significantly greater than that of Red's internal coupling. The measures now are shown in Figure 21, virtually identical to Figure 20, so the Red side has gained significantly: Blue is neither able to more smoothly track Red nor track more precisely ahead of Red; Red is not able to disrupt Blue any further in Blue's goal of fixing a certain point ahead of Red's OODA loop.

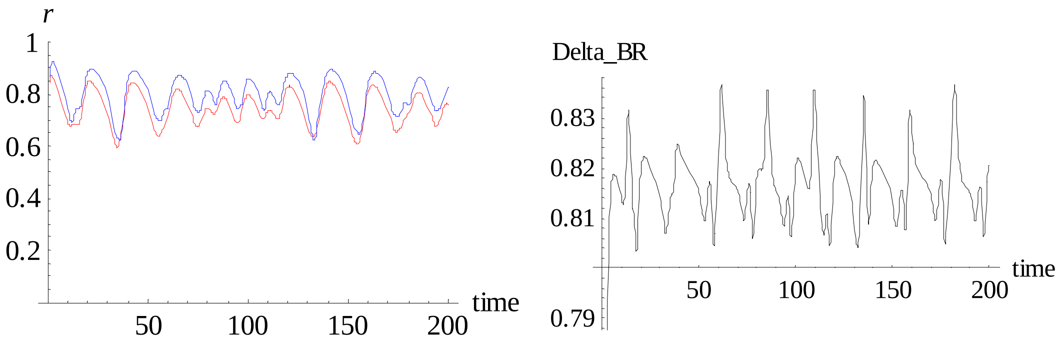

Figure 21 The *r* order parameter for Blue and Red (left) and Delta_BR (right) for case Eq. (21)

I have explored further increases in the Red-to-Blue coupling and changing the network structure of the Blue force, introducing a small number of random links between the 'tactical' level (the leaves of Figure 4(a)) and the 'operational' and 'strategic' levels (the middle and apex nodes). These scenarios are also indistinguishable from Figure 21. Blue is unable to perform any better because it remains, essentially, a hierarchy. Red is unable to perform better because it has limited ISR on Blue and is not seeking to operate 'inside the decision cycle' of Blue. In this intermediate coupling range, all of these scenarios generate cyclic behaviour of the measures: Blue achieves close to its intended state with respect to Red but with cyclic fluctuations away from this point.

It is noteworthy that, to achieve the ideal state of Blue quasi-locking ahead of Red, the coupling needs to be very high. This is a consequence of Blue's already high internal coupling (compared to that for Red to achieve the same state), and this in turn is needed to enable the hierarchical structure of the Blue C2 system to reach close to self-synchronisation. In words, hierarchically structured C2 systems can match flat networks but at the cost of tighter internal responsiveness and even tighter responsiveness to the adversary. On the other hand, the accountability that hierarchies traditionally support means they may be resourced to have superior ISR capabilities that give them more insight into the adversary's decision cycle than the adversary has into theirs.

These patterned behaviours are emergent in that they can neither be understood nor foreseen in terms of the Blue and Red systems as wholes, much less the individual oscillators. When all couplings are non-zero, one might easily anticipate either unadulterated success – locking of the two systems at some point in relation to each other – or some chaotic behaviour representing an irresolvable tug-of-war between adversaries. The patterned behaviour is explained retrospectively in terms of the intermediate scale structures of clusters and their interactions, none of which were intended in the design of the original networks (recall my comment on the absence of 'management' of the heterogeneity of the decision maker frequencies); they arise as an accident of the falling of the dice on individual frequencies across the network. Thus, in a certain range of couplings, the three layers of the system – the individual oscillators, the Red and Blue C2 networks separately and the entire interacting system – are supplemented by an intermediate layer of clusters of nodes in each C2 network, and another layer in which Red and Blue nodes (or clusters of nodes) may cluster with each other in their adversarial interactions.

6. Applications and Future Work

I have shown that the Kuramoto model of synchronising phase oscillators can be applied to model adversarial C2 systems, reflecting Boyd's principle that success follows from operating inside the adversary's decision loop. The resulting model exhibits rich dynamical behaviours. The illustrative example I selected, of a pure hierarchy opposing a random network, is intended as a caricature of a direct force on force, such as may be encountered in contemporary counter-insurgency operations. The method can also be used to pair-wise test opposing headquarters designs. Real data on network structure is straightforwardly applied into this model. Realistic frequency distributions should also be more sophisticated in that a multi-echelon C2 structure would not all operate on the same time scales: planning teams would work to longer time-frames, operators to shorter scales. The frequency distributions would thus generate nested cycles: operators seeking to synchronise with each other in loops that cycle inside slower planning cycles to which planners seek to synchronise. In such a case the interaction function requires some modification. Within any echelon, one may study the impact of greater training in reducing decision frequency heterogeneity – narrowing the frequency range of Blue compared to that of Red. I have also used a highly idealistic ISR model for Blue. A Blue force rarely has access to every stage of the decision-making cycle of Red. Narrower ISR functions over specific or even random points in the decision cycle can be represented. More challenging, but I believe achievable, is the task of representing memory or learning effects by replacing the sine function with longer tail functions. The model thus may incorporate a range of rich properties of C2 systems – all with a set of coupled differential equations solved on a desktop. It is not yet the definitive model of C2, but one that may genuinely bridge the chasm between static and vastly more complex dynamical C2 representations.

Acknowledgements

I am grateful for fruitful discussions with colleagues Richard Taylor, Tony Dekker and Mathew Zuparic; I thank Iain Macleod for a critical reading of the paper.

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